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GENERAL DECAY OF SOLUTION FOR A VISCOELASTIC-TYPE TRANSMISSION PROBLEM WITH VARYING DELAY

Safia Meftah[1*], **Fares Yazid**[2], **Fatima Siham Djeradi**[2]

[1]Laboratory of Operator Theory, EDP and Applications, University of Echahid hamma lakhdar-ElOued 39000, Algeria
[2]Laboratory of Pure and Applied Mathematics, Amar Telidji University-Laghouat 03000, Algeria

Abstract. This manuscript focuses on a transmission problem in a bounded domain with viscoelastic term acting on the first equation and varying delay terms. Under suitable assumptions on the weight of the damping and the weight of the delay, we prove the exponential stability of the solution by introducing a suitable Lyapunov functional.

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Corresponding author: Meftah, Safia, Laboratory of Operator Theory, EDP and Applications, University of Echahid hamma lakhdar-ElOued 39000, Algeria, e-mail: *safia-meftah@univ-eloued.dz*
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1 Introduction

In this article, we study the following transmission system with a viscoelastic term and varying delay term

$$\begin{cases} \phi_{tt}(x,t) - a\phi_{xx}(x,t) + \displaystyle\int_0^t g(t-s)\phi_{xx}(x,s)\mathrm{d}s \\ + \mu_1\phi_t(x,t) + \mu_2\phi_t(x,t-\tau(t)) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\ \quad \psi_{tt}(x,t) - b\psi_{xx}(x,t) = 0, \quad (x,t) \in (L_1,L_2) \times (0,+\infty), \end{cases} \tag{1}$$

under the boundary and transmission conditions

$$\begin{cases} \phi(0,t) = u(L_3,t) = 0, \\ \phi(L_i,t) = \psi(L_i,t), \quad i = 1,2, \\ a\phi_x(L_i,t) - \displaystyle\int_0^t g(t-s)\phi_x(L_i,t)\mathrm{d}s = b\psi_x(L_i,t), \quad i = 1,2, \end{cases} \tag{2}$$

and the initial conditions

$$\begin{cases} \phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x), \quad x \in \Omega, \\ \phi_t(x,t-\tau(t)) = f_0(x,t-\tau(t)), \quad x \in \Omega, \ t \in (0,\bar{\tau}), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad x \in (L_1,L_2), \end{cases} \tag{3}$$

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where $0 < L_1 < L_2 < L_3$, $\Omega = (0, L_1) \cup (L_2, L_3)$, a, b, μ_1, μ_2 are positive constants, and where the time varying delay $\tau(t) > 0$, satisfies

$$0 \leq \tau(t) \leq \bar{\tau}, \quad \forall t > 0, \tag{4}$$

$$\tau \in W^{2,\infty}(0,T), \ \forall T > 0, \tag{5}$$

and

$$\tau'(t) \leq d_0 < 1, \ \forall t > 0, \tag{6}$$

where, d_0 is a positive constant.

We are interested in proving the exponential stability of the problem (1)-(3). In order to obtain this, we will assume that

$$\max\left\{1, \frac{a}{b}\right\} < \frac{L_1 + L_3 - L_2}{2(L_2 - L_1)}, \tag{7}$$

As described in Benseghir (2003), the assumption (7) gives the relation between the boundary regions and the transmission permitted.

Time delay is a characteristic of a physical system that causes the response to an applied force to be delayed in its consequences. The main question is whether a system that is asymptotically stable in the absence of delays can become unstable due to their presence .

The design of material components is directly tied to transmission problems, which have received a lot of attention recently, for example in the analysis of damping mechanisms in the metallurgical industry or in smart materials technology.

Studies on transmission problems connected to (1)-(3) have also been explored. The transmission problem with frictional damping has been investigated by Bastos & Raposo (2007), the authors showed the wellposedness and exponential stability of the energy. The transmission of viscoelastic waves was a problem that Muñoz Rivera & Oquendo (2000) studied, they proved that no matter how tiny the size of the solution is, the dissipation caused by the viscoelastic portion can produce exponential decay of the solution. Bae (2010) considered the transmission problem, in which one component is clamped and the other is in a viscoelastic fluid producing a dissipative mechanism on the boundary, and established a decay result which depends on the rate of the decay of the relaxation function.

For the quasilinear problems, Cavalcanti et al. (2002) studied, in a bounded domain, the following equation :

$$|\phi_t|^\rho - \Delta\phi - \Delta\phi_{tt} + \int_0^t g(t-s)\Delta\phi(x)ds - \gamma\Delta\phi_t = 0, \quad in \quad \Omega \times \mathbb{R}_*^+,$$

for $\rho > 0$. They established a global existence result for $\gamma \geq 0$, and an exponential decay result for $\gamma > 0$.

The transmission problem with history and delay was invastigated by Li et al. (2016), where the equations were expressed as

$$\phi_{tt} - a\phi_{xx} + \int_0^{+\infty} g(s)\phi_{xx}(x, t-s)ds + \mu_1 u_t(x,t) + \mu_2\phi_t(x, t-\tau) = 0, \ \text{ in } \ \Omega\times]0, +\infty[,$$

$$\psi_{tt} - b\psi_{xx} = 0, \quad \text{ in } \quad]L_1, L_2[\times]0, +\infty[,$$

and proved an exponential stability result for two cases, under appropriate assumptions on function g and on the delay term. In the first case, they considered $\mu_2 < \mu_1$ and in the second case, they assumed that $\mu_2 = \mu_1$.

Messaoudi (2008) established a more general decay result, in a bounded domain of the following viscoelastic equation:

$$\phi_{tt} - \Delta\phi + \int_0^t g(t-\tau)\Delta\phi(\tau)d\tau = 0, \quad \text{in } \Omega \times (0,\infty).$$

After that, Han & Wang (2011) investigated the nonlinear viscoelastic equation:

$$\phi_{tt} - \Delta\phi + \int_0^t g(t-\tau)\Delta\phi(\tau)d\tau + |\phi|^k \partial j(\phi_t) = |\phi|^{p-1}\phi, \quad \text{in } \Omega \times (0,T),$$

the authors proved the global existence of generalized solutions, weak solutions for the equation.

To the best of our knowledge, A. Benseghir's contribution in Benseghir (2003) was the first one made in the literature for the transmission problem with a time delay. More specifically, the following transmission problem

$$\begin{cases} \phi_{tt} - a\phi_{xx} + \mu_1\phi_t(x,t) + \mu_2\phi_t(x,t-\tau) = 0, & \text{in} \quad \Omega \times]0,+\infty[, \\ \psi_{tt} - b\psi_{xx} = 0, \quad \text{in} \quad]L_1, L_2[\times]0,+\infty[, \end{cases} \tag{8}$$

with constant weights $\mu_1,\ \mu_2$ and time delay $\tau > 0$ was studied. Under suitable assumption on the weights of the two feedbacks ($\mu_1 < \mu_2$), the author proved the well-posedness of the system, and established an exponential decay result under condition (7).

Wang et al. (2016) extended the finding from Benseghir (2003) and demonstrated the solution's existence and uniqueness using the Faedo-Galerkin approach, and its exponential stability using the energy method.

Inspired by the above results, in this study, we are interested in investigating the general decay result of problem (1)-(3) under some hypotheses. For asymptotic behavior, we establish a general decay result from which the exponential and polynomial types of decay are just specific cases, by constructing an appropriate Lyaponov functional.

The remainder of this paper is organized as follows. In section 2, we provide some resources used in our research, then highlight our main results. In section 3, we introduce some technical lemmas which are fundamental in the proof of our stability result. In section 4, we prove the decay result.

2 Preliminaries and main results

In this section, we provide some practical materials that are required to prove our main results. Let's first introduce the notations below:

$$\begin{aligned} (g \star \Phi)(t) &:= \int_0^t g(t-s)\Phi(s)\,ds, \\ (g\Diamond\Phi)(t) &:= \int_0^t g(t-s)|\Phi(t)-\Phi(s)|\,ds, \\ (g\Box\Phi)(t) &:= \int_0^t g(t-s)|\Phi(t)-\Phi(s)|^2\,ds. \end{aligned}$$

The above operators clearly satisfy

$$\begin{aligned} (g \star \Phi)(t) &:= \left(\int_0^t g(s)ds\right)\Phi(t) - (g\Diamond\Phi)(t), \\ |(g\Diamond\Phi)(t)|^2 &\le \left(\int_0^t |g(s)|ds\right)(|g|\Box\Phi)(t). \end{aligned}$$

Lemma 1 (Cavalcanti et al. (2003)). *The following equation holds, for any $g,\ \Phi \in C^1(\mathbb{R})$:*

$$2\left[g \star \Phi\right] h' = g' \Box \Phi - g(t)|\Phi|^2 - \frac{d}{dt}\left\{ g\Box\Phi - \left(\int_0^t g(s)ds\right)|\Phi|^2 \right\}.$$

Proof. We differentiate the expression

$$g\Box\Phi - \left(\int_0^t \Phi(s)ds\right)|\Phi|^2,$$

to find the result. □

The following assumptions apply to the relaxation function g:

(A_1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function which satisfy

$$g \in L^1(0,\infty), \quad g(0) > 0, \quad 0 < \beta(t) := a - \int_0^t g(s)\mathrm{d}s,$$

$$0 < \beta_0 := a - \int_0^\infty g(s)\mathrm{d}s.$$

(A_2) There exists a non-increasing differentiable function $\zeta(t) : \mathbb{R}^+ \to \mathbb{R}^+$, such that

$$g'(t) \leq -\zeta(t)g(t), \quad \forall t \geq 0 \quad \text{and} \quad \int_0^\infty \zeta(t)\mathrm{d}t = +\infty.$$

According to these hypotheses, we have

$$\beta_0 \leq \beta(t) \leq a. \tag{9}$$

Similar to Nicaise & Pignotti (2006), we introduce the following variable

$$z(x,p,t) = \phi_t(x, t - \tau(t)p), \quad (x,p,t) \in \Omega \times (0,1) \times (0,\infty),$$

so, the variable z satisfies

$$\tau(t)z_t(x,p,t) + (1 - \tau'(t)p)z_p(x,p,t) = 0, \quad (x,p,t) \in \Omega \times (0,1) \times (0,\infty).$$

Then, problem (1) can be rewritten as

$$\begin{cases} \phi_{tt}(x,t) - a\phi_{xx}(x,t) + g \star \phi_{xx} + \mu_1\phi_t(x,t) + \mu_2 z(x,1,t) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\ \psi_{tt}(x,t) - b\psi_{xx}(x,t) = 0, \quad (x,t) \in (L_1, L_2) \times (0,+\infty), \\ \tau(t)z_t(x,p,t) + (1 - \tau'(t)p)z_p(x,p,t) = 0, \quad (x,p,t) \in \Omega \times (0,1) \times (0,+\infty), \end{cases} \tag{10}$$

the boundary and transmission conditions (2) take the following form

$$\begin{cases} \phi(0,t) = \phi(L_3,t) = 0, \\ \phi(L_i,t) = \psi(L_i,t), \quad i = 1,2,\ t \in (0,+\infty), \\ \left(a - \int_0^t g(s)\mathrm{d}s\right)\phi_x(L_i,t) = b\psi_x(L_i,t), \quad i = 1,2,\ t \in (0,+\infty), \end{cases} \tag{11}$$

and the initial conditions (3) become

$$\begin{cases} \phi(x,t) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x),\ x \in \Omega, \\ z(x,0,t) = \phi_t(x,t), \quad z(x,1,t) = f_0(x, t - \tau(t)), \quad (x,t) \in \Omega \times (0,+\infty), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad x \in (L_1, L_2). \end{cases} \tag{12}$$

As in Raposo (2008), we introduce the Hilbert spaces

$$X_* = \Big\{(\phi,\psi) \in H^1(\Omega) \cap H^1(L_1,L_2) : \phi(0,t) = \phi(L_3,t) = 0, \phi(L_i,t) = \psi(L_i,t),$$
$$\left(a - \int_0^t g(s)\,ds\right)\phi_x(L_i,t) = b\psi_x(L_i,t), i = 1,2\Big\}$$

and

$$\mathbb{L}^2 = L^2(\Omega) \times L^2(L_1,L_2).$$

According to previous results in the literature (see Wang et al. (2016)), we state the following well-posedness result, which can be proved by using the Faedo–Galerkin method.

Theorem 1. *Assume that (A_1) and (A_2) hold. Then for $(\phi_0,\psi_0) \in X_*$, $(\phi_1,\psi_1) \in \mathbb{L}^2$, and $f_0 \in L^2((0,1),\Omega)$, problem (10)-(12) admits a unique weak solution (ϕ,ψ,z), such that*

$$(\phi,\psi) \in C((0,\infty);X_*) \cap C^1((0,\infty);\mathbb{L}^2),$$
$$z \in C((0,\infty);L^2((0,\infty),\Omega)).$$

Now, we shall continue and define the energy functional of the solution of problem (1)-(3) by

$$\mathscr{E}(t) = \frac{1}{2}\int_0^1 \left\{\phi_t^2(x,t) + \beta(t)\phi_x^2(x,t) + (g\Box\phi_x)\right\}dx + \frac{1}{2}\int_{L_1}^{L_2}\left\{\psi_t^2(x,t) + b\psi_x^2(x,t)\right\}dx \tag{13}$$
$$+\frac{\xi}{2}\tau(t)\int_0^1\int_0^1 z^2(x,p,t)dp\,dx.$$

where, ξ satisfies

$$\frac{\mu_2}{\sqrt{1-d_0}} < \xi < 2\mu_1 - \frac{\mu_2}{\sqrt{1-d_0}}. \tag{14}$$

Theorem 2. *Let (ϕ,ψ,z) be the solution of problem (1)-(3). Assume that (A_1), (A_2) and*

$$a > \frac{8(L_2-L_1)}{L_1+L_3-L_2}\beta_0, \quad b > \frac{8(L_2-L_1)}{L_1+L_3-L_2}\beta_0 \tag{15}$$

hold, then there exist constants κ_0, $\kappa_2 > 0$ such that, for all $t \in \mathbb{R}^+$ and for all $\kappa_1 \in (0,\kappa_0)$,

$$\mathscr{E}(t) \le \kappa_2 e^{-\kappa_1\int_0^t \zeta(s)ds}. \tag{16}$$

3 Technical lemmas

In this section, we study the asymptotic behavior of problem (1)-(3). We state and prove some technical lemmas which are essential in the proof of our stability result. We utilize multiplier technique to establish stability results for the energy of the solution of system(1). This necessitates creating an appropriate Lyapunov functional equivalent to energy as we clarify in the following section.

We prove the decay result, under assumptions (14) and

$$\mu_2 < \sqrt{1-d_0}\mu_1. \tag{17}$$

Lemma 2. *Let (ϕ,ψ,z) be the solution of problem (10)-(12). Then we have the following estimate*

$$\mathscr{E}'(t) \le -c_1\int_\Omega \phi_t^2(x,t)\,dx - c_2\int_\Omega z^2(x,1,t)\,dx + \frac{1}{2}\int_\Omega (g'\Box\phi_x)(t)\,dx. \tag{18}$$

Proof. We start by multiplying $(10)_1$ and $(10)_2$ by ϕ_t and ψ_t respectively, then, we integrate by parts and we use (11), to obtain

$$\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega [\phi_t^2(x,t)+a\phi_x^2(x,t)]\,dx\right\}+\frac{d}{dt}\left\{\int_{L_1}^{L_2}\left[\psi_t^2(x,t)+b\psi_x^2(x,t)\right]\,dx\right\}$$

$$=-\mu_1\int_\Omega \phi_t^2(x,t)\,dx-\mu_2\int_\Omega \phi_t(x,t)z(x,1,t)\,dx+\int_0^t g(t-s)\int_\Omega \phi_x(s)\phi_{xt}(t)\,dxds. \tag{19}$$

Using Lemma 1, we can rewrite the last term in the right-hand side of (19) as

$$\int_0^t g(t-s)\int_\Omega \phi_x(s)\phi_{xt}(t)\,dxds+\frac{1}{2}g(t)\int_\Omega \phi_x^2\,dx$$

$$=\frac{1}{2}\frac{d}{dt}\left\{\int_0^t g(s)\int_\Omega \phi_x^2\,dxds-\int_\Omega (g\Box\phi_x)(t)\,dx\right\}+\frac{1}{2}\int_\Omega (g'\Box\phi_x)(t)\,dx.$$

then, (19) becomes

$$\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega [\phi_t^2(x,t)+a\phi_x^2(x,t)]\,dx\right\}+\frac{d}{dt}\left\{\int_{L_1}^{L_2}\left[\psi_t^2(x,t)+b\psi_x^2(x,t)\right]\,dx\right\}+\frac{1}{2}\int_\Omega (g\Box\phi_x)(t)\,dx$$

$$=-\mu_1\int_\Omega \phi_t^2(x,t)\,dx-\mu_2\int_\Omega \phi_t(x,t)z(x,1,t)\,dx-\frac{1}{2}g(t)\int_\Omega \phi_x^2\,dx+\frac{1}{2}\int_\Omega (g'\Box\phi_x)(t)\,dx. \tag{20}$$

Next, we multiply the last equation in (10) by ξz and integrate the result over $\Omega\times(0,1)$ with respect to x and p, respectively, we obtain

$$\begin{aligned}\frac{\xi}{2}\frac{d}{dt}\int_\Omega\int_0^1 \tau(t)z^2(x,p,t)dp\,dx=&-\xi\int_\Omega\int_0^1 (1-\tau'(t)p)z(x,p,t)z_p(x,p,t)dp\,dx\\&+\frac{\xi}{2}\tau'(t)\int_\Omega\int_0^1 z^2(x,p,t)dp\,dx\\=&-\frac{\xi}{2}\int_\Omega\int_0^1 \frac{\partial}{\partial_p}(1-\tau'(t)p)z^2(x,p,t)dp\,dx\\=&\frac{\xi}{2}\int_\Omega \left(z^2(x,0,t)-z^2(x,1,t)\right)dx+\frac{\xi\tau'(t)}{2}\int_\Omega z^2(x,1,t)\,dx.\end{aligned} \tag{21}$$

Now, using (20) and (21), we get

$$\begin{aligned}\frac{d}{dt}\mathscr{E}(t)=&-\left(\mu_1-\frac{\xi}{2}\right)\int_\Omega \phi_t^2(x,t)\,dx-\frac{\xi}{2}\int_\Omega (1-\tau'(t))z^2(x,1,t)\,dx-\mu_2\int_\Omega \phi_t(x,t)z(x,1,t)\,dx\\&-\frac{1}{2}g(t)\int_\Omega \phi_x^2\,dx+\frac{1}{2}\int_\Omega (g'\Box\phi_x)(t)\,dx.\end{aligned} \tag{22}$$

Young's inequality in (22) gives

$$\begin{aligned}\frac{d}{dt}\mathscr{E}(t)\le&-\left(\mu_1-\frac{\xi}{2}-\frac{\sqrt{1-d_0}\,\mu_2}{2}\right)\int_\Omega \phi_t^2(x,t)\,dx-\left(\frac{\xi}{2}(1-d_0)+\frac{\mu_2\sqrt{1-d_0}}{2}\right)\int_\Omega z^2(x,1,t)\,dx\\&+\frac{1}{2}\int_\Omega (g'\Box\phi_x)(t)\,dx.\end{aligned}$$

Finally, we exploit (17) to complete the proof. □

Lemma 3. *Let (ϕ, ψ, z) be the solution of problem (10)-(12). The functional $\mathscr{R}$ defined by*

$$\mathscr{R}(t) := \int_\Omega \phi\phi_t \mathrm{d}x + \frac{\mu_1}{2}\int_\Omega \phi^2 \mathrm{d}x + \int_{L_1}^{L_2} \psi\psi_t \mathrm{d}x,$$

satisfies

$$\begin{aligned}\frac{d}{dt}\mathscr{R}(t) \leq & \int_\Omega \phi_t^2 \mathrm{d}x + \int_{L_1}^{L_2} \psi_t^2 \mathrm{d}x + (L^2\epsilon + \epsilon - \beta(t))\int_\Omega \phi_x^2 \mathrm{d}x - \int_{L_1}^{L_2} b\psi_x^2 \mathrm{d}x \\ & + \frac{1}{4\epsilon}(a - \beta(t))\int_\Omega (g\Box\phi_x)\, dx + \frac{\mu_2^2}{4\epsilon}\int_\Omega z^2(x,1,t)\mathrm{d}x.\end{aligned} \tag{23}$$

Proof. Differentiating $\mathscr{R}$ and using (10), we get

$$\frac{d}{dt}\mathscr{R}(t) = \int_\Omega \phi_t^2 \mathrm{d}x - \int_\Omega (a\phi_x - g\star\phi_x)\phi_x\, dx - \mu_2\int_\Omega z(x,1,t)\phi\, dx + \int_{L_1}^{L_2} \psi_t^2\, dx - \int_{L_1}^{L_2} b\psi_x^2\, dx$$

$$= \int_\Omega \phi_t^2 \mathrm{d}x - \beta(t)\int_\Omega \phi_x^2 \mathrm{d}x - \int_\Omega (g\Diamond\phi_x)\phi_x\, dx - \mu_2\int_\Omega z(x,1,t)\phi\, dx + \int_{L_1}^{L_2} \psi_t^2\, dx - \int_{L_1}^{L_2} b\psi_x^2\, dx. \tag{24}$$

Using the boundary conditions (2), we get

$$\phi^2(x,t) = \left(\int_0^x \phi_x(x,t)dx\right)^2 \leq L_1\int_0^{L_1} \phi_x^2(x,t)dx, \quad x\in[0,L_1],$$

$$\phi^2(x,t) \leq (L_3 - L_2)\int_{L_2}^{L_3} \phi_x^2(x,t)dx, \quad x\in[L_2,L_3],$$

Which indicates

$$\int_\Omega \phi^2(x,t)dx \leq L^2\int_\Omega \phi_x^2(x,t)dx, \quad x\in\Omega, \quad \text{where} \quad L = \max\{L_2, L_3 - L_2\}. \tag{25}$$

exploiting (25) and applying Young's and Poincare's inequalites, we find for any $\epsilon > 0$,

$$\mu_2\int_\Omega z(x,1,t)\phi\, dx \leq \frac{\mu_2^2}{4\epsilon}\int_\Omega z^2(x,1,t)\mathrm{d}x + L^2\epsilon\int_\Omega \phi_x^2 \mathrm{d}x. \tag{26}$$

Again, Young's inequality and (A_1) give

$$\int_\Omega (g\Diamond\phi_x)\phi_x\, dx \leq \epsilon\int_\Omega \phi_x^2\, dx + \frac{1}{4\epsilon}\int_\Omega (g\Diamond\phi_x)^2\, dx \leq \epsilon\int_\Omega \phi_x^2\, dx + \frac{1}{4\epsilon}(a-\beta(t))\int_\Omega (g\Box\phi_x)\, dx. \tag{27}$$

We obtain the desired outcome by inserting the estimates (26) and (27) into (24). □

Inspired by Marzocchi et al. (2002), we consider the function

$$q(x) = \begin{cases} x - \frac{L_1}{2}, & x\in[0,L_1], \\ \frac{L_1}{2} - \frac{L_1+L_3-L_2}{2(L_2-L_1)}(x - L_1), & x\in(L_1,L_2), \\ x - \frac{L_2+L_3}{2}, & x\in[L_2,L_3]. \end{cases} \tag{28}$$

The fact that $q(x)$ is bounded is obvious, since $|q(x)| \leq M$, where $M = \max\{\frac{L_1}{2}, \frac{L_3-L_2}{2}\}$ is a positive constant.

Lemma 4. *Let (ϕ, ψ, z) be the solution of problem (10)-(12). The functional $\mathscr{I}_1$ defined by*

$$\mathscr{I}_1(t) = -\int_\Omega q(x)\phi_t(a\phi_x - g \star \phi_x)\mathrm{d}x, \tag{29}$$

satisfies, for any $\epsilon_1 > 0$,

$$\begin{aligned}\frac{d}{dt}\mathscr{I}_1(t) \leq &\left[-\frac{q(x)}{2}(a\phi_x - g\star\phi_x)^2\right]_{\partial\Omega} - \left[\frac{a}{2}q(x)\phi_t^2\right]_{\partial\Omega} + \left[\frac{a}{2} + \frac{\mu_1^2}{2\epsilon_1} + \frac{M^2}{4\epsilon_1}\right]\int_\Omega \phi_t^2 dx \\ &+ \left[\epsilon_1 M^2 a^2 + \beta^2(t) + 2M^2\epsilon_1(a-\beta(t))^2 + c_5^2\epsilon_1\right]\int_\Omega \phi_x^2 dx + \frac{\mu_2^2}{2\epsilon_1}\int_\Omega z^2(x,1,t)\,dx \\ &+(1+2M^2\epsilon_1)(a-\beta(t))\int_\Omega (g\Box\phi_x)\,dx + (a-\beta(t))\epsilon_1\int_\Omega (g'\Box\phi_x)\,dx. \end{aligned}\tag{30}$$

Proof. Differentiating $\mathscr{I}_1(t)$ and using (10), we find

$$\begin{aligned}\frac{d}{dt}\mathscr{I}_1(t) =& -\int_\Omega q(x)\phi_{tt}(a\phi_x - g\star\phi_x)\,dx - \int_\Omega q(x)\phi_t(a\phi_{xt} - g(t)\phi_x(t) + (g'\diamondsuit\phi_x)(t))\,dx \\ =& \left[-\frac{q(x)}{2}(a\phi_x - g\star\phi_x)^2\right]_{\partial\Omega} + \frac{1}{2}\int_\Omega q'(x)(a\phi_x - g\star\phi_x)^2\,dx - \left[\frac{a}{2}q(x)\phi_t^2\right]_{\partial\Omega} \\ &+\frac{a}{2}\int_\Omega q'(x)\phi_t^2\,dx - \int_\Omega q(x)(\mu_1\phi_t(x,t) + \mu_2 z(x,1,t))(g\star\phi_x)\,dx \\ &+\int_\Omega q(x)a\phi_x(\mu_1\phi_t(x,t)+\mu_2 z(x,1,t))\,dx - \int_\Omega q(x)\phi_t\left[(g'\diamondsuit\phi_x)(t) - g(t)\phi_x\right]\,dx. \end{aligned}\tag{31}$$

We see that

$$\begin{aligned}\frac{1}{2}\int_\Omega q'(x)(a\phi_x - g\star\phi_x)^2\,dx &= \frac{1}{2}\int_\Omega\left[\left(a - \int_0^t g(s)ds\right)\phi_x + g\diamondsuit\phi_x\right]^2\,dx \\ \leq \int_\Omega |\beta(t)|^2\phi_x^2\,dx + \int_\Omega |g\diamondsuit\phi_x|^2\,dx &\leq \int_\Omega |\beta(t)|^2\phi_x^2\,dx + (a-\beta(t))\int_\Omega (g\Box\phi_x)\,dx. \end{aligned}\tag{32}$$

By Young's inequality, we find for any $\epsilon_1 > 0$,

$$\int_\Omega q(x)a\phi_x(\mu_1\phi_t(x,t)+\mu_2 z(x,1,t))\,dx \leq \epsilon_1 M^2 a^2\int_\Omega \phi_x^2\,dx + \frac{\mu_1^2}{4\epsilon_1}\int_\Omega \phi_t^2\,dx + \frac{\mu_2^2}{4\epsilon_1}\int_\Omega z^2(x,1,t)\,dx, \tag{33}$$

$$\begin{aligned}&\int_\Omega q(x)(\mu_1\phi_t(x,t) + \mu_2 z(x,1,t))(g\star\phi_x)\,dx \\ &\leq \epsilon_1 M^2\int_\Omega (g\star\phi_x)^2\,dx + \frac{\mu_1^2}{4\epsilon_1}\int_\Omega \phi_t^2\,dx + \frac{\mu_2^2}{4\epsilon_1}\int_\Omega z^2(x,1,t)\,dx, \\ &\leq 2\epsilon_1 M^2(a-\beta(t))^2\int_\Omega \phi_x^2\,dx + 2\epsilon_1 M^2(a-\beta(t))\int_\Omega (g\Box\phi_x)\,dx + \frac{\mu_1^2}{4\epsilon_1}\int_\Omega \phi_t^2\,dx + \frac{\mu_2^2}{4\epsilon_1}\int_\Omega z^2(x,1,t)\,dx, \end{aligned}\tag{34}$$

and

$$\int_\Omega q(x)\phi_t\left[(g'\diamondsuit\phi_x)(t) - g(t)\phi_x\right]\,dx \leq \frac{M^2}{4\epsilon_1}\int_\Omega \phi_t^2\,dx + c_5\epsilon_1\int_\Omega \phi_x^2\,dx + (a-\beta(t))\epsilon_1\int_\Omega (g'\Box\phi_x)\,dx. \tag{35}$$

We obtain (30), by inserting (32)-(35) into (31).

□

Lemma 5. *Let (ϕ, ψ, z) be the solution of problem (10)-(12). The functional $\mathscr{I}_2$ defined by*

$$\mathscr{I}_2(t) = -\int_{L_1}^{L_2} q(x)\psi_x\psi_t \mathrm{d}x, \tag{36}$$

satisfies the estimate

$$\frac{d}{dt}\mathscr{I}_2(t) \leq -\frac{L_1+L_3-L_2}{4(L_2-L_1)}\left(\int_{L_1}^{L_2}\psi_t^2\, dx + \int_{L_1}^{L_2}\psi_x^2\, dx\right) + \frac{L_1}{4}\psi_t^2(L_1) + \frac{L_3-L_2}{4}\psi_t^2(L_2)$$
$$+\frac{b}{4}((L_3-L_2)\psi_x^2(L_2,t) + L_1\psi_x^2(L_1,t)). \tag{37}$$

Proof. Using the same procedure, taking the derivative of $\mathscr{I}_2(t)$ with respect to t, we get

$$\begin{aligned}
\frac{d}{dt}\mathscr{I}_2(t) &= -\int_{L_1}^{L_2} q(x)\psi_{xt}\psi_t\, dx - \int_{L_1}^{L_2} q(x)\psi_x\psi_{tt}\, dx\\
&= \left[-\frac{q(x)}{2}\psi_t^2\right]_{L_1}^{L_2} + \frac{1}{2}\int_{L_1}^{L_2} q'(x)\psi_t^2\, dx + \frac{1}{2}\int_{L_1}^{L_2} bq'(x)\psi_x^2\, dx + \left[\frac{-bq(x)}{2}\psi_x^2\right]_{L_1}^{L_2}\\
&\leq -\frac{L_1+L_3-L_2}{4(L_2-L-1)}\left(\int_{L_1}^{L_2}\psi_t^2\, dx + \int_{L_1}^{L_2} b\psi_x^2\, dx\right) + \frac{1}{4}L_1\psi_t^2(L_1) + \frac{L_3-L_2}{4}\psi_t^2(L_2)\\
&+ \frac{1}{4}b((L_3-L_2)\psi_x^2(L_2,t) + L_1\psi_x^2(L_1,t)).
\end{aligned}$$

Which concludes the proof. □

Similar to **?**, we introduce the following functional

$$\mathscr{I}_3(t) = \xi\tau(t)\int_\Omega\int_0^1 e^{-2\tau(t)\,p}z^2(x,p,t)dp\, dx,\ t \geq 0.$$

Lemma 6. *Let (ϕ, ψ, z) be the solution of problem (10)-(12). Then the functional $\mathscr{I}_3$ satisfies,*

$$\frac{d}{dt}\mathscr{I}_3(t) \leqslant -2\mathscr{I}_3(t) + \xi\int_\Omega \phi_t^2 dx. \tag{38}$$

Proof. Differentiating $\mathscr{I}_3(t)$, we find

$$\begin{aligned}
\frac{d}{dt}\mathscr{I}_3(t) =&\xi\tau'(t)\int_\Omega\int_0^1 e^{-2\tau(t)\,p}z^2(x,p,t)\, dpdx\\
&-2\xi\tau(t)\tau'(t)\int_\Omega\int_0^1 e^{-2\tau(t)\,p}p\, z^2(x,p,t)\, dpdx\\
&+2\xi\tau(t)\int_\Omega\int_0^1 e^{-2\tau(t)\,p}z(x,p,t)z_t(x,p,t)\, dpdx,
\end{aligned} \tag{39}$$

now, using the last equation in (10), we get

$$\tau(t)\int_\Omega\int_0^1 e^{-2\tau(t)\,p}zz_t\, dp\, dx = \int_0^1\int_0^1 e^{-2\tau(t)\,p}(\tau'(t)p-1)zz_p\, dpdx, \tag{40}$$

notice that

$$\begin{aligned}
\int_\Omega\int_0^1 e^{-2\tau(t)\,p}(\tau'(t)p-1)(zz_p)(x,p,t)\, dp\, dx =& \frac{1}{2}\int_\Omega\int_0^1 \frac{\partial}{\partial p}\left(e^{-2\tau(t)\,p}(\tau'(t)p-1)z^2(x,p,t)\right) dpdx\\
&+\tau(t)\int_\Omega\int_0^1 e^{-2\tau(t)\,p}(\tau'(t)p-1)z^2(x,p,t)\, dpdx\\
&-\frac{\tau'(t)}{2}\int_\Omega\int_0^1 e^{-2\tau(t)\,p}z^2(x,p,t)\, dpdx.
\end{aligned} \tag{41}$$

Using (40) and (41), equation (39) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\mathscr{I}_3(t) = &-2\xi\tau(t)\int_\Omega\int_0^1 e^{-2\tau(t)\,p}z^2(x,p,t)\,dpdx + \xi\int_\Omega \phi_t^2(x,t)\,dx\\ &-\xi(1-\tau'(t))e^{-2\tau(t)\,p}\int_\Omega z^2(x,1,t)\,dx,\end{aligned}$$

As a result, estimate (38) directly follows. □

4 Decay of solutions

In this section, we prove our main stability results, using the lemmas mentioned in Section 3.

Proof of Theorem 2. We define the Lyapunov functional

$$L(t) = N_1\mathscr{E}(t) + N_2\mathscr{R}(t) + N_3\mathscr{I}_1(t) + N_4\mathscr{I}_2(t) + \mathscr{I}_3(t), \tag{42}$$

where N_1, N_2, N_3, N_4 are later-fixed positive constants.

We proceede by taking the derivative of (42) with respect to t, then, we use estimates (18), (23), (30), (37), and (38), to obtain

$$\begin{aligned}\frac{d}{dt}L(t) \le &-\Big\{N_1c_1 - \xi - N_2 - N_3\Big(\frac{a}{2}+\frac{\mu_1^2}{2\epsilon_1}+\frac{M^2}{4\epsilon_1}\Big)\Big\}\int_\Omega \phi_t^2\mathrm{d}x\\ &-\Big\{N_1c_2 - \frac{\mu_2^2N_2}{4\epsilon} - N_3\frac{\mu_2^2}{2\epsilon_1}\Big\}\int_\Omega z^2(x,1,t)\mathrm{d}x\\ &-\big\{N_2(\beta(t)-L^2\epsilon-\epsilon) - N_3\left(\epsilon_1M^2a^2+\beta^2(t)+2M^2\epsilon_1(a-\beta(t))^2+c_5^2\epsilon_1\right)\big\}\int_\Omega \phi_x^2\mathrm{d}x\\ &-\Big\{\frac{b(L_1+L_3-L_2)}{4(L_2-L_1)}N_4 + N_2b\Big\}\int_{L_1}^{L_2}\psi_x^2\mathrm{d}x\\ &-\Big\{\frac{L_1+L_3-L_2}{4(L_2-L_1)}N_4 - N_2\Big\}\int_{L_1}^{L_2}\psi_t^2\mathrm{d}x\\ &-(bN_3-N_4)\frac{b}{4}\left((L_3-L_2)\psi_x^2(L_2,t)+L_1\psi_x^2(L_1,t)\right)\\ &-(aN_3-N_4)\Big[\frac{L_1}{4}\psi_t^2(L_1,t)+\frac{L_3-L_2}{4}\psi_t^2(L_2,t)\Big]\\ &+c(N_2,N_3)\int_\Omega(g\Diamond\phi_x)\mathrm{d}x\\ &+\left(\frac{N_1}{2}-c(N_3)\right)\int_\Omega(g'\Box\phi_x)\mathrm{d}x.\end{aligned} \tag{43}$$

Now, we select our coefficients in (43), carefully, in a way that all the coefficients in (43) will be negative. Indeed under (15), we can find N_2, N_3 and N_4 such that

$$N_2 < \frac{L_1+L_3-L_2}{4(L_2-L_1)N_4}, \quad N_4 > bN_3, \quad N_2 > 2N_3\beta_0.$$

After fixing the above constants, we can choose ϵ and ϵ_1 small enough such that

$$N_2(L^2\epsilon+\epsilon) + N_3(\epsilon_1M^2a^2 + 2M^2\epsilon_1(a-\beta(t))^2 + c_5\epsilon_1) < N_2 - N_3\beta(t).$$

Choosing N_1 to be large enough so that

$$\begin{cases} N_1c_1 - \xi - N_2 - N_3\Big(\frac{a}{2}+\frac{\mu_1^2}{2\epsilon_1}+\frac{M^2}{4\epsilon_1}\Big) > 0,\\ N_1c_2 - \frac{\mu_2^2N_2}{4\epsilon} - N_3\frac{\mu_2^2}{2\epsilon_1} > 0,\\ \frac{N_1}{2} - c(N_3) > 0,\end{cases}$$

we conclude that, there exist two positive constants γ_1 and γ_2 such that (43) takes the following form

$$\frac{d}{dt}L(t) \leq -\gamma_1\mathscr{E}(t) + \gamma_2 \int_\Omega (g\Box\phi_x)dx. \tag{44}$$

Yet, according to functionals $\mathscr{R}(t)$, $\mathscr{I}_1(t)$, $\mathscr{I}_2(t)$, $\mathscr{I}_3(t)$ and $\mathscr{E}(t)$ definition, for sufficiently large N_1, there exists a positive constant γ_3, fulfilling

$$|N_2\mathscr{R}(t) + N_3\mathscr{I}_1(t) + N_4\mathscr{I}_2(t) + \mathscr{I}_3(t)| \leq \gamma_3\mathscr{E}(t),$$

which indicates that

$$(N_1 - \gamma_3)\mathscr{E}(t) \leq L(t) \leq (N_1 + \gamma_3)\mathscr{E}(t).$$

Now, we shall estimate the last term in (43).
Using (A_2) and (18), we obtain

$$\zeta(t)\int_\Omega (g\Box\phi_x)dx \leq \int_\Omega [(\zeta g)\Box\phi_x]\ dx \leq -\int_\Omega (g'\Box\phi_x)\ dx \leq -2\frac{d}{dt}\mathscr{E}(t). \tag{45}$$

At this point, we introduce the functional

$$\mathscr{L}(t) = \zeta(t)L(t) + 2\gamma_2\mathscr{E}(t).$$

Given (A_2) and the fact that $L(t)$ and $\mathscr{E}(t)$ are equivalent, then, for some positive constants η_1 and η_2, we have

$$\eta_1\mathscr{E}(t) \leq \mathscr{L}(t) \leq \eta_2\mathscr{E}(t), \tag{46}$$

From (45), (46) and (A_2), we find

$$\begin{aligned}
\frac{d}{dt}\mathscr{L}(t) &= \zeta'(t)L(t) + \zeta(t)\frac{d}{dt}L(t) + 2\gamma_2\frac{d}{dt}\mathscr{E}(t) \\
&\leq \zeta(t)\left(-\gamma_1\mathscr{E}(t) + \gamma_2\int_\Omega (g\Box\phi_x)dx\right) + 2\gamma_2\frac{d}{dt}\mathscr{E}(t) \\
&\leq -\zeta(t)\gamma_1\mathscr{E}(t) \\
&\leq -\kappa_0\zeta(t)\mathscr{L}(t),
\end{aligned}$$

where $\kappa_0 = \frac{\gamma_1}{\eta_2}$. We deduce that, for any $\kappa_1 \in (0, \kappa_0)$,

$$\frac{d}{dt}\mathscr{L}(t) \leq \mathscr{L}(0)e^{-\kappa_1\int_0^t \zeta(s)ds}, \quad \text{for any } \ t > 0. \tag{47}$$

Finally, (16) is established using (46) and (47). This concludes the proof of Theorem 2. □

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